THE

MATHEMATICAL GAZETTE.

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LONDON:
GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN,
AND BOMBAY.

VOL. IV. JANUARY, 1908.—PART I. No. 68.

SOLUTIONS.

608. [L[1]. 14. a.] *A parallelogram circumscribes the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *and two of its corners move on* $Ax^2+2Hxy+By^2=1$. *Shew that the other two move on* $$Bb^4x^2-2Ha^2b^2xy+Aa^4y^2=b^2x^2+a^2y^2-a^2b^2.$$

The tangents at α, β to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet at the point
$$a.\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)},\quad b.\frac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)},$$
and those at $\pi+\alpha$, β at the point
$$a.\frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)},\quad -b.\frac{\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)},$$
or putting $\frac{1}{2}(\alpha+\beta)=\theta$, $\frac{1}{2}(\alpha-\beta)=\phi$, these points are
$$\left(\frac{a\cos\theta}{\cos\phi},\ \frac{b\sin\theta}{\cos\phi}\right)\text{ and }\left(\frac{a\sin\theta}{\sin\phi},\ -\frac{b\cos\theta}{\sin\phi}\right).$$
The first of these lies on the given conic;
$$\therefore\ Aa^2\cos^2\theta+2Hab\sin\theta\cos\theta+Bb^2\sin^2\theta=\cos^2\phi.$$
Hence, calling the second (x, y), it satisfies
$$Aa^2.\frac{y^2}{b^2}-2Hab.\frac{xy}{ab}+Bb^2.\frac{x^2}{a^2}=\cot^2\phi=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$
as given.

609. [K. 2. a.] *If O be any point on the circumcircle of a triangle ABC, and if OA, OB, OC meet the opposite sides in a, b, c respectively, shew that the straight line on which lie the points of intersection of ab and AB, bc and BC, ca and CA, passes through a point which is independent of the position of O.*

The point O being $(\alpha', \beta', \gamma')$, the point a is $(0, \beta', \gamma')$. Hence the equation to bc is
$$\begin{vmatrix}\alpha, & \beta, & \gamma\\ \alpha', & 0, & \gamma'\\ \alpha', & \beta', & 0\end{vmatrix}=0\quad\text{or}\quad -\frac{\alpha}{\alpha'}+\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'}=0,$$

and the intersection of this with $a=0$ lies on

$$\frac{a}{a'}+\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'}=0,$$

and so for the other two.

Also, since O is on the circumcircle, the line last written passes through the fixed point (a, b, c).

610. [R. 4. a.] *A regular tetrahedron of height h has a tetrahedron of height xh cut off by a plane parallel to the base. When the remaining frustum is placed on one of its slant faces on a horizontal plane, it is just on the point of falling over. Shew that x is a root of the equation*

$$x^3+x^2+x=2.$$

Let O be the common vertex, A and A' the centres of the bases. The volumes of the tetrahedra are as $1:x^3$ and the distances of their centres of gravity from O are $\frac{3}{4}h$, $\frac{3}{4}xh$. Hence the distance of G, the C. of G. of the frustum, from O is

$$\bar{x}=\frac{\frac{3}{4}h-\frac{3}{4}x^4h}{1-x^3}=\frac{3}{4}h.\frac{(1+x)(1+x^2)}{1+x+x^2}.$$

Hence $$A'G=\bar{x}-xh=\frac{3-(x+x^2+x^3)}{4(1+x+x^2)}.h \text{......................(i)}$$

Let AN be the $\perp^r$ from O on the edge of the base in contact with the ground, and let ON cut the other edge in N'. Then by the question, GN' must be vertical. Now the edge of the larger tetrahedron is $\frac{\sqrt{3}}{\sqrt{2}}.h$;

$$\therefore\ AN=\frac{1}{3}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{\sqrt{2}}\cdot h=\frac{1}{2\sqrt{2}}h;\quad \therefore\ A'N'=\frac{1}{2\sqrt{2}}.xh.$$

But ONG' is a right angle;

$$\therefore\ A'G.xh=\left(\frac{1}{2\sqrt{2}}.xh\right)^2,\quad i.e.\ A'G=\tfrac{1}{8}xh. \text{ ....(ii)}$$

Equating the values in (i) and (ii), we get the required equation.

611. [R. 9. b. a.] *Two equal balls A and B are placed on an inclined plane and start from rest at the same instant. After the lower ball B has passed over a space b it impinges perpendicularly on a plane (coefficient of elasticity e), and after rebounding comes to rest just as it strikes A. Prove that*

$$\frac{a}{b}=2e(e+1),$$

where a is the initial distance between the balls.

The striking velocity of B is $\sqrt{2g\sin a.b}$, and $\therefore$ its velocity after impact is $e\sqrt{2g\sin a.b}$; $\therefore$ after impact B describes a distance up the plane

$$\frac{e^2(2g\sin a.b)}{2g\sin a}=e^2b.$$

Also the total time B is in motion is $(1+e)\sqrt{\frac{2b}{g\sin a}}$, and in this time A describes a distance

$$\tfrac{1}{2}g\sin a.(1+e)^2.\frac{2b}{g\sin a}=b(1+e)^2.$$

Hence $$b(1+e)^2+e^2b=a+b,$$
$$i.e.\ a=b(2e+2e^2).$$

612. [L¹. 7. c.] *P is a point on a hyperbola, PN its ordinate, M a point on the conjugate axis such that $AM=CP$. Prove that if S be a focus,*

$$CM : PN = CS : CB,$$

and if P', M' be a corresponding pair of points, $PM'=P'M$.

We have $$CN^2+PN^2=CA^2+CM^2.$$
But $$PN^2 : CN^2-CA^2=CB^2 : CA^2;$$
$$\therefore\ PN^2 : PN^2+CN^2-CA^2=CB^2 : CS^2,$$
$$i.e.\ PN : CM=CB : CS.$$
Again, $$\left.\begin{aligned} PM'^2&=CP^2+CM'^2-2CM'.PN \\ P'M^2&=CP'^2+CM^2-2CM.P'N' \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$
But $$PN : CM=P'N' : CM';\quad \therefore\ CM'.PN=CM.P'N'.$$
Also $$CP^2-CM^2=AM^2-CM^2=CA^2=CP'^2-CM'^2.$$
Hence, from (i), subtracting, we see that $PM'^2-P'M^2=0$.

613. [D. 2. b.] *Sum the series*

$$\text{(i)}\ x^n-n(x+y)^n+\frac{n(n-1)}{2!}(x+2y)^n-\ldots.$$

$$\text{(ii)}\ \frac{3}{1.3}+\frac{7}{2.4}.3+\frac{11}{3.5}.3^2+\frac{15}{4.6}.3^3+\ldots.$$

each to $(n+1)$ terms.

(i) The given series is $n!$ times the coeff. of t^n in the expansion

$$e^{xt}-n.e^{(x+y)t}+\frac{n(n-1)}{2!}e^{(x+2y)t}-\ldots$$
$$=e^{xt}(1-e^{yt})^n=(1+xt+\ldots)\left(-yt-\frac{y^2t^2}{2!}-\ldots\right)^n,$$

in which the coeff. of t^n is $(-y)^n$.

Hence the value of the given series is $(-1)^n.n!.y^n$.

(ii) The nth term is $$\frac{4n-1}{n(n+2)}.3^{n-1}=\left[\frac{9}{2(n+2)}-\frac{1}{2n}\right].3^{n-1}$$
$$=\frac{1}{2}\left(\frac{3^{n+1}}{n+2}-\frac{3^{n-1}}{n}\right).$$

Hence the sum of $(n+1)$ terms is

$$\frac{1}{2}\left(\frac{3^{n+2}}{n+3}-\frac{3}{2}-1\right)=\frac{1}{2}.\frac{3^{n+2}}{n+3}-\frac{5}{4}.$$

614. [K. 20. d.] *Shew that if $\cos(\alpha+\theta)=\cos\alpha\cos\phi-\cos\beta\sin\alpha\sin\phi$, where θ and ϕ are small, then θ is very approximately equal to*

$$\phi\cos\beta+\tfrac{1}{2}\phi^2\cot\alpha\sin^2\beta.$$

Retaining small quantities of the second order, the given relation is

$$(1-\tfrac{1}{2}\theta^2)\cos\alpha-\theta.\sin\alpha=(1-\tfrac{1}{2}\phi^2)\cos\alpha-\phi.\cos\beta\sin\alpha\ldots\ldots\ldots\ldots\text{(i)}$$

Now suppose $\theta=A\phi+B\phi^2$. Then (i) becomes

$$\cos\alpha-A\sin\alpha.\phi-(\tfrac{1}{2}A^2\cos\alpha+B\sin\alpha)\phi^2$$
$$=\cos\alpha-\cos\beta\sin\alpha.\phi-\tfrac{1}{2}\cos\alpha.\phi^2.$$

This must be an identity. Hence

$$A\sin\alpha=\cos\beta\sin\alpha;\qquad \therefore\ A=\cos\beta.$$
$$\tfrac{1}{2}A^2\cos\alpha+B\sin\alpha=\tfrac{1}{2}\cos\alpha;\qquad \therefore\ B=\tfrac{1}{2}\cot\alpha\sin^2\beta.$$

615. [K. 11. d.] *Pairs of circles are drawn having external contact with each other and also with the circles*

$$(x-a)^2+y^2=r^2, \quad (x-ma)^2+y^2=m^2r^2.$$

Shew that the points in which the pairs of circles touch each other lie on the circle

$$x^2+y^2=m(a^2-r^2).$$

Let the centre of one of the touching circles be (α, β) and its radius ρ.

Then $\quad (\alpha-a)^2+\beta^2=(r+\rho)^2$ ......(i), $\quad (\alpha-ma)^2+\beta^2=(mr+\rho)^2$,

whence subtracting, $\quad 2(a\alpha+r\rho)=(m+1)(a^2-r^2)$;

$\therefore$ from (i) $\quad \alpha^2+\beta^2-\rho^2=m(a^2-r^2)$

and similarly for the second circle, $\alpha'^2+\beta'^2-\rho'^2=m(a^2-r^2)$ ....(ii)

Hence the radical axis of the two circles passes through the origin. But since they touch, this means that the common tangent at the point of contact passes through the origin. But from (ii), the square of the tangent from the origin to either circle is $m(a^2-r^2)$. Hence if (x, y) be the point of contact,

$$x^2+y^2=m(a^2-r^2).$$

616. [L[1]. 10. a.] *Three tangents to a parabola form a triangle PQR, and the circle PQR cuts the parabola in the four points A, B, C, D. Prove that, if S be the focus, then*

$$\frac{SA \,.\, SB \,.\, SC \,.\, SD}{SP \,.\, SQ \,.\, SR}$$

is constant.

The equation of the circle through the intersection of tangents at α, β, γ on $\frac{l}{r}=1+\cos\theta$ is

$$r=\frac{l}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}\,.\cos\left(\theta-\frac{\alpha+\beta+\gamma}{2}\right).$$

Hence at the intersection of the circle and parabola we have

$$\left(kr-\frac{l-r}{r}\cos\lambda\right)^2=\frac{2rl-l^2}{r^2}\sin^2\lambda,$$

where $\quad k=2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\Big/l, \quad \lambda=\frac{\alpha+\beta+\gamma}{2}.$

This equation is a quartic in r, the product of the roots being $\frac{l^2}{k^2}$.

Also $\quad SP=\frac{l}{2\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$, etc. $\quad \therefore SP \,.\, SQ \,.\, SR=\frac{l}{2k^2}.$

Hence the value of the given expression is $2l$.

617. [R. 7. b.] *Prove that the maximum range on a horizontal plane of missiles projected from a hill of height h with velocity V is*

$$\frac{V^2}{g}\left(1+\frac{2gh}{V^2}\right)^{\frac{1}{2}}.$$

If h be 300 ft. and V be 1,000 ft. per sec., find over what distance a similar battery on the plane would be under fire, and unable to return it.

If α is the angle of elevation, x the range on the horizontal plane, we have

$$-h = x\tan\alpha - \tfrac{1}{2}g \cdot \frac{x^2}{V^2}(1+\tan^2\alpha).$$

If this quadratic in $\tan\alpha$ has real roots,

$$x^2 > 4 \cdot \tfrac{1}{2}g \cdot \frac{x^2}{V^2}\left(\tfrac{1}{2}g \cdot \frac{x^2}{V^2} - h\right),$$

$$\text{whence } x^2 < \frac{V^2}{g^2}(V^2+2gh),$$

$$i.e.\ x < \frac{V^2}{g}\left(1+\frac{2gh}{V^2}\right)^{\frac{1}{2}}.$$

For the battery on the plane, to find the greatest distance from the foot of the hill at which the other can be hit we have evidently only to change the sign of h in this result. Hence the distance over which the battery is unable to return fire is

$$\frac{V^2}{g}\left[\left(1+\frac{2gh}{V^2}\right)^{\frac{1}{2}} - \left(1-\frac{2gh}{V^2}\right)^{\frac{1}{2}}\right]. \quad \text{.............................(i)}$$

With the given values $\frac{2gh}{V^2} = \cdot 0192$, *i.e.* a small fraction. Hence, by the Binomial Theorem, the distance (i) is approximately equal to

$$\frac{V^2}{g} \cdot \frac{2gh}{V^2} = 2h,$$

i.e. the distance required is about 600 feet.

618. [K. 12. a.] *Find the locus of a point at which two given portions of the same straight line subtend equal angles.*

If P be the point, the bisectors of the angle APD will be also those of the angle BPC. If these are PE, PF, then E, F are harmonic conjugates for both A, D and B, C. To find these points draw the circles XAD, XBC, where X is any point not in AB. Let these intersect again in Y, and let XY cut AB in R (which will be outside the circles). Draw a tangent RZ to either circle, and with centre R and radius RZ draw a circle cutting AB in E, F. These are the points required, since evidently

$$RE^2 = RF^2 = RA \cdot RD = RB \cdot RC,$$

and the locus of P is the circle on EF as diameter.

619. [D. b. γ. ; D. 2. d.] *Shew that*

$$(2\cos\theta - 1)(2\cos 2\theta - 1)\ldots(2\cos 2^{n-1}\theta - 1) \equiv \frac{2\cos 2^n\theta + 1}{2\cos\theta + 1},$$

and evaluate the continued fraction

$$\cot\theta - \frac{2}{\cot 2\theta -}\,\frac{2}{\cot 2^2\theta -}\ldots - \frac{2}{\cot 2^{n-1}\theta -}\,\frac{2}{\tan 2^n\theta}.$$

(i) $(2\cos\theta + 1)(2\cos\theta - 1) = 4\cos^2\theta - 1 = 2\cos 2\theta + 1,$

$(2\cos 2\theta + 1)(2\cos 2\theta - 1) = 2\cos 2^2\theta + 1,$

....

$(2\cos 2^{n-1}\theta + 1)(2\cos 2^{n-1}\theta - 1) = 2\cos 2^n\theta + 1.$

Multiplying these we obtain the given result

(ii) We have, from the identity,

$$\cot a - \frac{2}{\tan 2a} = \tan a,$$

$$\cot 2^{n-1}\theta - \frac{2}{\tan 2^n\theta} = \tan 2^{n-1}\theta,$$

$$\cot 2^{n-2}\theta - \frac{2}{\tan 2^{n-1}\theta} = \tan 2^{n-2}\theta,$$

....

$$\cot \theta - \frac{2}{\tan 2\theta} = \tan \theta.$$

From these it follows that the value of the given C.F. is $\tan\theta$.

620. [L[1]. 10. a.] *The tangents at the points Q and R of the parabola $y^2=4ax$ intersect at P. The perpendicular from P on QR meets the axis at G, and S is the focus. Prove that the radius of the circle PQR is*

$$SP.PG \div 2a.$$

If P is the point (x', y'), the equation to the circle PQR must be of the form

$$y^2 - 4ax + \lambda(2ax - yy' + 2ax')(2ax + yy' + k) = 0,$$

with the condition $\lambda . 4a^2 = 1 - \lambda y'^2$, *i.e.* $\lambda = 1/(4a^2 + y'^2)$.

Also, since the circle passes through (x', y'),

$$\therefore\ -1 + \lambda(2ax' + y'^2 + k) = 0,$$

whence $k = 4a^2 - 2ax'$.

Substituting and reducing, we find the equation of the circle to be

$$a(x^2+y^2) - (y'^2 + 2a^2)x + y'(x'-a)y + ax'(2a - x') = 0,$$

whence, if R be its radius,

$$4a^2R^2 = (y'^2 + 2a^2)^2 + y'^2(x'-a)^2 - 4a^2x'(2a - x')$$
$$= \{y'^2 + (x'-a)^2\}(y'^2 + 4a^2).$$

Now the equation to PG is $xy' + 2ay = x'y' + 2ay'$, and this cuts the axis where $x = x' + 2a$. $\therefore\ PG^2 = 4a^2 + y'^2$.

Hence $$2a.R = SP.PG.$$

621. [L[1]. 5. a.] *Shew that the line joining the pole of a normal chord of the conic $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with its pole with regard to*

$$\frac{x^2}{a^2+n^2b^2} + \frac{y^2}{b^2+n^2a^2} = 1$$

touches the ellipse $$a^2x^2 + b^2y^2 = (a^2+b^2)^2.$$

If (x_1, y_1) is the pole of the chord normal at a, then the lines

$$\frac{xx_1}{a^2} + \frac{yy_1}{\beta^2} = 1, \text{ and } ax\sin a - by\cos a = (a^2 - b^2)\sin a\cos a$$

must coincide;

$$\therefore\ \frac{x_1}{a^2 a\sin a} = \frac{y'}{-\beta^2 b\cos a} = \frac{1}{(a^2-b^2)\sin a\cos a}.$$

Hence the chord joining the poles is

$$\begin{vmatrix} x & , & y & , & 1 \\ (a^2+nb^2)a\sin\alpha, & & -(b^2+na^2)b\cos\alpha, & & (a^2-b^2)\sin\alpha\cos\alpha \\ a^3\sin\alpha & , & -b^3\cos\alpha & , & (a^2-b^2)\sin\alpha\cos\alpha \end{vmatrix}=0.$$

Subtracting the third row from the second, this is

$$\begin{vmatrix} x, & y, & 1 \\ b\sin\alpha, & -a\cos\alpha, & 0 \\ a^3\sin\alpha, & -b^3\cos\alpha, & (a^2-b^2)\sin\alpha\cos\alpha \end{vmatrix}=0,$$

or $$ax\cos\alpha+by\sin\alpha=a^2+b^2,$$

which is a tangent to $a^2x^2+b^2y^2=(a^2+b^2)^2$.

622. [R. 7. b.] *Prove that the least velocity with which a particle must be projected from a point on the ground a feet in front of a wall h feet high in order to pass over the wall is*

$$\{g(h+\sqrt{h^2+a^2})\}^{\frac{1}{2}}.$$

If the velocity of projection is u, the maximum range on an inclined plane of angle β is $\dfrac{u^2}{g(1+\sin\beta)}$.

Hence β, being the angle which the line joining the point of projection to the top of the wall makes with the horizontal, we must have

$$\frac{u^2}{g(1+\sin\beta)}\nless\sqrt{h^2+a^2}.$$

But $$\sin\beta=\frac{h}{\sqrt{h^2+a^2}}\,;\quad\therefore\ u^2\nless g\left(1+\frac{h}{\sqrt{h^2+a^2}}\right)\sqrt{h^2+a^2},$$

$$i.e.\ u^2\nless g(\sqrt{h^2+a^2}+h).$$

623. [R. 9. b.] *A wedge of mass M and angle α rests on a smooth horizontal plane. A perfectly elastic particle of mass m impinges on it with velocity u. Shew that the particle rebounds with velocity*

$$\frac{M-m\sin^2\alpha}{M+m\sin^2\alpha}\cdot u,$$

assuming that no kinetic energy is lost by the impact.

Let u_1 be the velocity of rebound, v that of the wedge. Then by Newton's Law,

$$-u_1-v\sin\alpha=-u. \quad\text{.................................(i)}$$

Also since no K.E. is lost;

$$\therefore\ \tfrac{1}{2}Mv^2+\tfrac{1}{2}mu_1^2=\tfrac{1}{2}mu^2. \quad\text{..............................(ii)}$$

From (i) and (ii),

$$(M+m\sin^2\alpha)u_1^2-2Muu_1+(M-m\sin^2\alpha)u^2=0,$$

$$i.e.\ (u_1-u)[(M+m\sin^2\alpha)u_1-(M-m\sin^2\alpha)u]=0.$$

Now $u_1 \neq u$, since the plane is not fixed;

$$\therefore\ u_1=\frac{M-m\sin^2\alpha}{M+m\sin^2\alpha}\cdot u.$$

624. [D. 2. d.] *Prove that the product of the first n convergents of the continued fraction* $\frac{4}{3+}\frac{4}{3+}\frac{4}{3+}\dots$ *is*

$$\frac{5.4^n}{4^{n+1}+(-1)^n}.$$

If u_n be either the numerator or denominator of the nth convergent, we have

$$u_n=3u_{n-1}+4u_{n-2}.$$

Assuming $u_n=A\lambda^n$, this gives $\lambda^2-3\lambda-4=0$, *i.e.* $\lambda=4$ or -1;

$$\therefore\ u_n=A.4^n+B.(-1)^n.$$

Putting in initial values, we easily get

$$p_n=\tfrac{4}{5}[4^n-(-1)^n],\quad q_n=\tfrac{1}{5}[4^{n+1}+(-1)^n];$$

$$\therefore\ \frac{p_n}{q_n}=\frac{4[4^n+(-1)^{n-1}]}{4^{n+1}+(-1)^n}.$$

Hence the product of the first n convergents is $\dfrac{4^n(4+1)}{4^{n+1}+(-1)^n}$.

625. [K. 20. c.] *If* α, β, γ, δ *are solutions of the equation*

$$\cos 2x+a\cos x+b\sin x+c=0,$$

no two of which differ by a multiple of π, *prove that* $\alpha+\beta+\gamma+\delta$ *is a multiple of* 2π.

Putting $\tan\frac{x}{2}=t$, the equation becomes

$$\frac{1-6t^2+t^4}{(1+t^2)^2}+a.\frac{1-t^2}{1+t^2}+b.\frac{2t}{1+t^2}+c=0,$$

i.e.
$$(1-a+c)t^4+2bt^3+2(c-3)t^2+2bt+(1+a+c)=0.$$

The roots of this are $t_1=\tan\frac{\alpha}{2}$, etc., and $\Sigma t_1=\Sigma t_1t_2t_3$;

$$\therefore\ \tan\frac{\alpha+\beta+\gamma+\delta}{2}=0,\quad i.e.\ \frac{\alpha+\beta+\gamma+\delta}{2}=n\pi.$$

626. [L¹. 10. a.] *If the product of the tangents drawn from a point P to the parabola* $y^2=4ax$ *is equal to the product of the focal distance of P and the latus rectum, prove that the locus of P is the parabola*

$$y^2=4a(x+a).$$ [Trip. 1897.

The intersection of tangents at m, m' is amm', $a(m+m')$.

Hence
$$t_1^2=(amm'-am^2)^2+[a(m+m')-2am]^2$$
$$=a^2(m-m')^2(m^2+1).$$

Also
$$SP^2=(amm'-a)^2+[a(m+m')]^2=a^2(m^2+1)(m'^2+1).$$

Hence, if $t_1t_2=4a.SP$, we must have $(m-m')^2=4$.

$\therefore$ putting $m+m'=p$, the co-ordinates of P are

$$x=a.\frac{p^2-4}{4},\ y=ap,$$

and the locus is $y^2=4a(x+a)$.

627. [L[1]. 16. a.] *A chord of an ellipse subtends a right angle at the point on the ellipse of eccentric angle α. Shew that it passes through the fixed point*

$$\left(a\,.\frac{a^2-b^2}{a^2+b^2}\cos\alpha,\quad b\,.\frac{b^2-a^2}{b^2+a^2}\sin\alpha\right).$$

If the chords (α, β) and (α, γ) are at right angles,

$$\frac{1}{a^2}\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}+\frac{1}{b^2}\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}=0,$$

i.e.
$$\frac{1}{a^2}\left[\cos\left(\alpha+\frac{\beta+\gamma}{2}\right)+\cos\frac{\beta-\gamma}{2}\right]+\frac{1}{b^2}\left[\cos\frac{\beta-\gamma}{2}-\cos\left(\alpha+\frac{\beta+\gamma}{2}\right)\right]=0,$$

or
$$\frac{a^2-b^2}{a^2+b^2}\cos\left(\alpha+\frac{\beta+\gamma}{2}\right)=\cos\frac{\beta-\gamma}{2}.$$

But this is precisely the condition that the point whose coordinates are given should lie on the chord (β, γ). Hence the result.

628. [L[1]. 17. e.] *Conics are drawn having the origin for a common focus, and touching the conic* $\frac{l}{r}=1+e\cos\theta$ *at the point* $\theta=\alpha$. *Prove that the least eccentricity of any such conic is*

$$e\sin\alpha(1+2e\cos\alpha+e^2)^{-\frac{1}{2}},$$

and that the latus-rectum of this conic is

$$2l(1+e\cos\alpha)(1+2e\cos\alpha+e^2)^{-1}.$$

Let one of the conics be $\frac{l'}{r}=1+e'\cos(\theta-\gamma)$. The tangents at α to the two conics are

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha),$$

$$\frac{l'}{r}=e'\cos(\theta-\gamma)+\cos(\theta-\alpha).$$

If these coincide,

$$\frac{e'\cos\gamma+\cos\alpha}{e+\cos\alpha}=\frac{e'\sin\gamma+\sin\alpha}{\sin\alpha}=\frac{l'}{l}, \qquad \text{..............................(i)}$$

whence
$$e'=\frac{e\sin\alpha}{\cos\gamma\sin\alpha-\sin\gamma(e+\cos\alpha)},$$

and the greatest value of the denominator is

$$\sqrt{\sin^2\alpha+(e+\cos\alpha)^2}=\sqrt{1+2e\cos\alpha+e^2}.$$

Further, from (i), putting $\frac{l'}{l}=\lambda$, we have

$$e'^2=[\lambda(e+\cos\alpha)-\cos\alpha]^2+[\lambda\sin\alpha-\sin\alpha]^2,$$
$$=\lambda^2(1+2e\cos\alpha+e^2)-2\lambda(1+e\cos\alpha)+1,$$

whence substituting for e', the result follows.

629. [R. 9. b.] *Two equal spheres, centres A and B, lie in contact on a smooth table, and A is struck directly by a third sphere, centre C, moving with velocity V in a direction making an acute angle θ with AB. Shew that after impact A moves in a direction making an angle*

$$\cot^{-1}(\cot\theta+2\tan\theta)$$

with CA produced.

Let V_1 be the velocity of C after impact, V_2 and V_3 those of A along and perpendicular to AB. Then B will move in the direction AB with velocity V_2. Hence the equations of momentum are

$$(V-V_1)\cos\theta=2V_2, \quad (V-V_1)\sin\theta=V_3.$$

Hence, if the direction of motion of A makes an angle ϕ with AB,

$$\tan\phi=\frac{V_3}{V_2}=2\tan\theta;$$

$$\therefore \tan(\phi-\theta)=\frac{\tan\theta}{1+2\tan^2\theta}=\frac{1}{\cot\theta+2\tan\theta}.$$

Hence the angle required is $\cot^{-1}(\cot\theta+2\tan\theta)$.

630. [R. 7. b. γ.] *Two particles are projected simultaneously from a point A so as to pass through another point B, the velocity of projection in each case being V. If α, α' are the angles of projection, prove that the particles will pass through B at times separated by the interval*

$$\frac{2V}{g}\cdot\frac{\sin\frac{1}{2}(\alpha-\alpha')}{\cos\frac{1}{2}(\alpha+\alpha')}.$$

Let t, t' be the times taken to reach B. Then we have

$$V\cos\alpha\,.\,t=V\cos\alpha'\,.\,t',$$

$$V\sin\alpha\,.\,t-\tfrac{1}{2}gt^2=V\sin\alpha'\,.\,t'-\tfrac{1}{2}gt'^2,$$

whence
$$\frac{t}{\cos\alpha'}=\frac{t'}{\cos\alpha}=k, \text{ say};$$

$$\therefore V\sin(\alpha-\alpha')=\tfrac{1}{2}gk(\cos^2\alpha'-\cos^2\alpha);$$

$$\therefore k=\frac{2V}{g}\cdot\frac{\sin(\alpha-\alpha')}{\cos^2\alpha'-\cos^2\alpha}=\frac{2V}{g}\cdot\frac{1}{\sin(\alpha+\alpha')};$$

$$\therefore t-t'=k(\cos\alpha'-\cos\alpha)$$

$$=\frac{2V}{g}\cdot\frac{\cos\alpha'-\cos\alpha}{\sin(\alpha+\alpha')}=\frac{2V}{g}\cdot\frac{\sin\frac{1}{2}(\alpha-\alpha')}{\cos\frac{1}{2}(\alpha+\alpha')}.$$

631. [D. 2. d.] *A continued fraction of the form $\frac{1}{p+}\frac{1}{q+}\frac{1}{r+}\dots$ has a, b, c for three successive quotients. A, B, C and A', B', C' are the coefficients of a, b, c in the numerator and denominator of any subsequent convergent. Prove that*

$$\frac{AC'-A'C}{AB'-A'B}+\frac{AC'-A'C}{BC'-B'C}$$

is constant, and determine its value.

Let $\frac{p}{q}$, $\frac{p'}{q'}$ be the two convergents immediately preceding the quotient a.

Then any subsequent convergent is given by

$$\frac{P}{Q}=\frac{\left(a+\frac{1}{b}+\frac{1}{c+x}\right)p+p'}{\left(a+\frac{1}{b}+\frac{1}{c+x}\right)q+q'},$$

where $c+x$ is the complete quotient up to this convergent.

Hence $$P=(abc+a+c+\overline{ab+1}\,x)p+(bc+bx+1)p',$$

and similarly for Q.

Thus
$$A=(bc+bx+1)p',$$
$$B=(ca+ax)p+(c+x)p'=(c+x)(ap+p'),$$
$$C=(ab+1)p+bp'.$$

Hence, taking $pq'-p'q=1$, we have
$$AC'-A'C=bc+bx+1,$$
$$BC'-B'C=ab(c+x)-(ab+1)(c+x)=-(c+x),$$
$$AB'-A'B=(bc+bx+1)(c+x).$$

Hence the value of the given expression is

$$\frac{1}{c+x}-\frac{bc+bx+1}{c+x}=-b.$$

632. [**K. 8. b.**] *If a, b, c, d be the lengths of the sides of a quadrilateral of area Δ inscribed in a circle of radius R, prove that*

$$16R^2\Delta^2=(bc+ad)(ca+bd)(ab+cd).$$

Denote the diagonals AC, BD by x, y. Let $\triangle ABC=\Delta_1$, $\triangle ADC=\Delta_2$.

Then $$4R\Delta_1=abx,\quad 4R\Delta_2=cdx;$$
$$\therefore\ 4R\Delta=(ab+cd)x,$$

i.e. $$16R^2\Delta^2=(ab+cd)^2.\frac{(bc+ad)(ac+bd)}{ab+cd}.$$

633. [**K. 2. e.**] *O is the centre of a circle of radius a. P, Q, R are any three points, Δ' the area of the triangle formed by their polars with regard to the circle, Δ the area of the triangle PQR and Δ_1, Δ_2, Δ_3 the areas of the triangles QOR, ROP, POQ. Prove that*

$$4\Delta'\Delta_1\Delta_2\Delta_3=a^4\Delta^2.$$

Let the equation to the circle be $x^2+y^2=a^2$. Then the polar of (x_1, y_1) is

$$xx_1+yy_1=a^2.$$

$$\therefore\ 2\Delta'=\frac{\begin{vmatrix} x_1, & y_1, & -a^2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}^2}{\begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix}\begin{vmatrix} x_2, & y_2 \\ x_3, & y_3 \end{vmatrix}\begin{vmatrix} x_3, & y_3 \\ x_1, & y_1 \end{vmatrix}}.$$

Now the numerator is $a^4.\begin{vmatrix} x_1, & y_1, & 1 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}^2 = a^4.(2\Delta)^2$;

$$\therefore \quad 2\Delta' = \frac{a^4.(2\Delta)^2}{(2\Delta_1)(2\Delta_2)(2\Delta_3)},$$

$$\text{i.e. } 4\Delta'\Delta_1\Delta_2\Delta_3 = a^4\Delta^2.$$

634. [**K. 8. a.**] *Prove that the middle points of the diagonals of the quadrilateral formed by the four straight lines*

$$\alpha=0, \ \beta=0, \ \gamma=0, \ \frac{\alpha}{l}+\frac{\beta}{m}+\frac{\gamma}{n}=0$$

lie on the straight line

$$al(b\beta+c\gamma-a\alpha)+bm(c\gamma+a\alpha-b\beta)+cn(a\alpha+b\beta-c\gamma)=0.$$

Let the line $\frac{\alpha}{l}+\frac{\beta}{m}+\frac{\gamma}{n}=0$ meet the sides of the triangle of reference in D, E, F. Then the co-ordinates of D are

$$\alpha=0, \ \frac{\beta}{m}=\frac{\gamma}{-n}=\frac{2\Delta}{bm-cn}.$$

Hence the middle point of AD is

$$\frac{\Delta}{a}, \ \frac{\Delta m}{bm-cn}, \ \frac{-\Delta n}{bm-cn}.$$

Similarly, the middle point of BE is

$$\frac{-\Delta l}{cn-al}, \ \frac{\Delta}{b}, \ \frac{\Delta n}{cn-al},$$

and the line joining these is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ bm-cn, & am, & -an \\ -bl, & cn-al, & bn \end{vmatrix}=0,$$

reducing to the given form. The symmetry of the result shews that the middle point of CF also lies on the line.

635. [**R. 4. d. α.**] *AB and BC are two uniform rods of the same material, freely jointed at B. The ends A and C are fixed in the same vertical. Prove (graphically or otherwise) that the stress at the joint is*

$$\tfrac{1}{2}W.\frac{BD}{AC},$$

where BD is the bisector of the angle ABC, and W the weight of the rods.

If we replace the weight of the rod AB by two forces at A and B, each equal to half the weight, the rod AB is in equilibrium under two forces at A and two at B, and the resultants of these pairs of forces must be equal and opposite and both in the line AB. Hence we construct the force diagram as follows: Draw ab, bc vertically downwards to represent the weights of AB, BC respectively. Bisect ab, bc in the points 1 and 2. Draw $1O$ and $2O$ parallel to AB, CB. Then Ob represents the reaction at B.

But
$$\frac{1b}{b2}=\frac{AB}{BC}=\frac{10}{20};$$

$\therefore$ Ob bisects the angle $1O2$.

Also
$$\frac{R}{W}=\frac{Ob}{ac}=\frac{1}{2}\cdot\frac{Ob}{12}=\frac{1}{2}\cdot\frac{BD}{AC}.$$

636. [R. 7. b. γ.] *A man h feet high fires from the level of his head a pistol, the ball from which issues with a velocity of V feet per second, just clears a wall a feet high, and strikes the ground as far beyond the wall as the man is from it. Shew that the pistol was held at an elevation*

$$sin^{-1}\left\{\frac{4a-3h}{2V}\sqrt{\frac{g}{2a-h}}\right\}.$$

The equation to the path referred to horizontal and vertical axes through the point of projection is

$$y=x\tan\alpha-\tfrac{1}{2}g\cdot\frac{x^2}{V^2\cos^2\alpha}.$$

Let the distance of the wall from the man be b. Then the path passes through the points $(a-h, b)$ and $(-h, 2b)$;

$$\therefore\ a-h=b\tan\alpha-\tfrac{1}{2}g\cdot\frac{b^2}{V^2\cos^2\alpha},$$

$$-h=2b\tan\alpha-\tfrac{1}{2}g\cdot\frac{4b^2}{V^2\cos^2\alpha},$$

whence
$$4a-3h=2b\tan\alpha,\quad 2a-h=g\cdot\frac{b^2}{V^2\cos^2\alpha}.$$

These give the required value of $\sin\alpha$.

637. [J. 1. c.] *Shew that the sum of all the homogeneous products of a, b, c of all dimensions from 0 to n is*

$$\Sigma\frac{a^{n+3}}{(a-b)(a-c)(a-1)}-\frac{1}{(a-1)(b-1)(c-1)}.$$

$$H_r=\text{coefficient of } x^r \text{ in } \frac{1}{(1-ax)(1-bx)(1-cx)}$$

$$=\ldots\ldots\ldots\ldots\ \Sigma\frac{a^2}{(a-b)(a-c)}\cdot\frac{1}{1-ax}\ \ldots\ldots\ldots\ldots\text{(i)}$$

$$=\sum_{a,b,c}\frac{a^{r+2}}{(a-b)(a-c)}.$$

Now
$$\sum_0^n a^{r+2}=\frac{a^2(a^{n+1}-1)}{a-1}.$$

Hence the sum required is

$$\Sigma\frac{a^2(a^{n+1}-1)}{a-1}\cdot\frac{1}{(a-b)(a-c)}$$

$$=\Sigma\frac{a^{n+3}}{(a-b)(a-c)(a-1)}-\Sigma\frac{a^2}{(a-b)(a-c)}\cdot\frac{1}{a-1}.$$

The latter sum, as is seen by putting $x=1$ in (i), is

$$\frac{1}{(a-1)(b-1)(c-1)}.$$

638. [K. 2. e.] *From G, the centre of gravity of a triangle ABC, perpendiculars GP, GQ, GR are let fall on the sides. Shew that*

(i) *the area of the triangle PQR is* $\frac{1}{18}(a^2+b^2+c^2)\sin A\sin B\sin C$;

(ii) *the radius of the circle PQR is* $\frac{4}{3}AD\,.\,BE\,.\,CF/(a^2+b^2+c^2)$, *where AD, BE, CF are the medians;*

(iii) *the sum of the areas of the circles PGQ, QGR, RGP is*

$$\frac{\pi}{12}(a^2+b^2+c^2).$$

(i) We have $GP=\frac{1}{3}b\sin C$, etc.;

$$\therefore\ \triangle PQR=\Sigma\triangle GQR=\tfrac{1}{2}\,.\,\Sigma GQ\,.\,GR\sin A$$

$$=\tfrac{1}{18}\,.\,\Sigma a\sin C\,.\,a\sin B\,.\,\sin A=\tfrac{1}{18}\,.\,\Sigma a^2\,.\,\sin A\sin B\sin C.$$

(ii) Also $\dfrac{PQ}{\sin A}=GC=\frac{2}{3}CF.$ Hence $\left(\text{from formula } R=\dfrac{abc}{4\Delta}\right)$,

radius of $\odot PQR=\dfrac{\frac{8}{27}AD\,.\,BE\,.\,CF\,.\sin A\sin B\sin C}{\frac{2}{9}\,.\,\Sigma a^2\,.\sin A\sin B\sin C}=\frac{4}{3}\,.\,\dfrac{AD\,.\,BE\,.\,CF}{\Sigma a^2}.$

(iii) Sum of areas of circles (since their diameters are AG, etc.)

$$=\tfrac{1}{4}\pi\,.\,\Sigma AG^2=\tfrac{1}{9}\pi\,.\,\Sigma AD^2=\tfrac{1}{12}\pi\,.\,\Sigma a^2.$$

639. [L^1. 4. a.] *Shew that the inclinations θ to the axis of x of the tangents drawn from the point (p, q) to the conic* $ax^2+by^2=1$ *are determined by the equation*

$$(ap^2+bq^2-1)(a+b\tan^2\theta)=(ap+bq\tan\theta)^2,$$

and also that the square of the length of the tangents whose inclination θ is thus determined is

$$\frac{1+\tan^2\theta}{a+b\tan^2\theta}(ap^2+bq^2-1).$$

The line $\dfrac{x-p}{\cos\theta}=\dfrac{y-q}{\sin\theta}=r$ meets the conic, where

$$r(a\cos^2\theta+b\sin^2\theta)+2r(ap\cos\theta+bq\sin\theta)+ap^2+bq^2-1=0.$$

If the line is a tangent, this quadratic in r has equal roots;

$$\therefore\ (ap\cos\theta+bq\sin\theta)^2=(a\cos^2\theta+b\sin^2\theta)(ap^2+bq^2-1).$$

Also in this case, each of the values of r is the length of the tangent, and the square of the tangent is the product of the roots,

$$\text{i.e.}\quad (ap^2+bq^2-1)/(a\cos^2\theta+b\sin^2\theta).$$

640. [R. 9. a.] *A rod of length 2a inclined at an angle θ to the horizontal rests tangentially against a fixed rough cylinder with its axis horizontal, being held in position by a horizontal string attached to its highest point and perpendicular to the axis of the cylinder. Find the friction at the point of contact, and prove that the part of the rod above the point of contact must lie between the limits*

$$a\cos\theta(\cos\theta\pm\mu\sin\theta),$$

where μ is the coefficient of friction.

Let A be the highest point of the rod, B the point of contact, G the middle point and O the centre of the sphere. Let the horizontal through A and the vertical through G meet in N. Then BN must be the direction of the friction at B; suppose it makes an acute angle ϕ with the radius. Then if

$$AB=x,$$

$$\frac{x}{AN}=\frac{\cos(\theta-\phi)}{\cos\phi},\quad \frac{AN}{a}=\cos\theta;\ \therefore\ \frac{x}{a}=\cos\theta(\cos\theta+\sin\theta\tan\phi).\ \text{......(i)}$$

Also, if F is the friction, $F=\dfrac{W}{\cos(\theta-\phi)}$, where ϕ is given by (i).

In limiting equilibrium $\phi=\pm\lambda$, the angle of friction;

$$\therefore \text{ from (i), } \frac{x}{a}=\cos\theta(\cos\theta\pm\mu\sin\theta).$$

641. [R. 7. b. γ.] *A regular hexagon stands with one side on the ground, and a particle is projected so as just to graze the four upper corners. Shew that the velocity of the particle on reaching the ground is to its least velocity as* $\sqrt{31}:\sqrt{3}$.

The equation to the path may be written in the form

$$\left(x-\frac{u^2\sin\alpha\cos\alpha}{g}\right)^2=-\frac{2u^2\cos^2\alpha}{g}\left(y-\frac{u^2\sin^2\alpha}{2g}\right).$$

From symmetry, the co-ordinates of the two lower corners are

$$\left(\frac{u^2\sin\alpha\cos\alpha}{g}\pm a,\ \frac{\sqrt{3}a}{2}\right),$$

and of the two upper corners,

$$\left(\frac{u^2\sin\alpha\cos\alpha}{g}\pm\frac{a}{2},\ \sqrt{3}a\right).$$

Hence
$$a^2=-\frac{2u^2\cos^2\alpha}{g}\left(\sqrt{3}\cdot\frac{a}{2}-\frac{u^2\sin^2\alpha}{2g}\right),$$
$$\frac{a^2}{4}=-\frac{2u^2\cos^2\alpha}{g}\left(\sqrt{3}a-\frac{u^2\sin^2\alpha}{2g}\right).$$

From these, $u^2\sin^2\alpha=\dfrac{7}{\sqrt{3}}ga$, $u^2\cos^2\alpha=\dfrac{\sqrt{3}}{4}ga$;

$$\therefore\ u^2=\frac{31}{4\sqrt{3}}\cdot ga;$$

$$\therefore\ u^2:u^2\cos^2\alpha=31:3.$$

642. [L[1]. 5. a.] *In an ellipse PSP' is a focal chord and the normal at P meets the minor axis in g. If V is the middle point of PP', prove that Vg is parallel to the tangent at P'.*

Draw the ordinates PN, $P'N'$. The lines PA, $A'P'$ will meet the directrix in the same point. Let this be K. Then, with the usual notation,

$$PN:AN=KX:AX \text{ and } P'N':A'N'=KX:A'X;$$

$$\therefore\ PN:P'N'=(1+e)AN:(1-e)A'N'.$$

Also $PN^2=(1-e^2)AN.NA'$; $\therefore\ PN.P'N'=(1-e)^2.A'N.A'N'$
$$=(1+e)^2.AN.AN' \text{ similarly};$$

$$\therefore\ 4e.PN.P'N'=(1-e^2)^2.2a(CN+CN'),$$

i.e. $$PN.P'N':SX^2=CN+CN':2CX=2a-PP'.2a. \quad\text{.........(i)}$$

Now draw $gE\ \perp^r$ to SP, and let the tangents at P, P' meet in Z.
Then $PE=a$, and $gE:a=SP:SZ=PN:SX$.
Also $VE=a-\frac{1}{2}PP'$; $\therefore$ by (i), $gE:VE=SX:P'N'=SZ:SP'$.
Hence the triangles VEg, ZSP' are similar, and Vg is $\parallel$ to ZP'.

643. [**K. 2. e.**] *If p, q, r be the lengths of the bisectors of the angles of a triangle produced to meet the circumcircle, and u, v, w the lengths of the perpendiculars of the triangle produced to meet the same circle, prove that*

$$p^2(v-w)+q^2(w-u)+r^2(u-v)=0.$$

If O is the orthocentre, and AO meets BC in L, and the circle in E, then

$$OL=LE;$$

$$\therefore\ u=AL+OL=2R\sin B\sin C+2R\cos B\cos C$$
$$=2R\cos(B-C).$$

Again let the bisector of the angle A meet the circle in D, and let DD' be a diameter. Then $\angle A\hat{D'}D=A\hat{B}D=B+\frac{A}{2}$;

$$\therefore\ p=2R\sin\left(B+\frac{A}{2}\right)=2R\cos\frac{B-C}{2};$$

$$\therefore\ \Sigma p^2(v-w)=8R^3\,.\,\Sigma\cos^2\frac{B-C}{2}(\cos\overline{C-A}-\cos\overline{A-B})$$
$$=4R^3\,.\,\Sigma(1+\cos\overline{B-C})(\cos\overline{C-A}-\cos\overline{A-B})$$
$$\equiv 0.$$

644. [**K. 20. c.**] *If the equation $\cot(\theta-a_1)+\cot(\theta-a_2)+\cot(\theta-a_3)=0$ has solutions θ_1, θ_2, θ_3 not differing by multiples of two right angles prove that $\theta_1+\theta_2+\theta_3-a_1-a_2-a_3$ is an odd multiple of a right angle.*

Let $\tan\theta=t$, $\tan\theta_1=t_1$, etc. Then the equation is

$$\Sigma\frac{1+tt_1}{t-t_1}=0,$$

i.e.
$$\Sigma(1+tt_1)(t-t_2)(t-t_3)=0,$$

or
$$t^3s_1+t^2(3-2s_2)+t(3s_3-2s_1)+s_2=0,$$

where $s_1=\Sigma t_1$, etc.

$$\therefore\ \tan(\theta_1+\theta_2+\theta_3)=\frac{\dfrac{2s_2-3}{s_1}+\dfrac{s_2}{s_1}}{1-\dfrac{3s_3-2s_1}{s_1}}$$
$$=\frac{s_2-1}{s_1-s_3}=-\cot(a_1+a_2+a_3);$$

$$\therefore\ \theta_1+\theta_2+\theta_3-a_1-a_2-a_3 \text{ is an odd multiple of } \frac{\pi}{2}.$$

645. [**L¹. 10. a.**] *Tangents are drawn from a given point (h, k) to a system of confocal and co-axial parabolas. Shew that the normals at the points of contact intersect on the line*

$$hx+ky+h^2+k^2=0.$$

The tangents at m, m' to $y^2=4ax$ intersect at amm', $a(m+m')$ and the normals at
$$a(m^2+mm'+m'^2)+2a,\quad -amm'(m+m').$$

Transferring to the focus as origin, the intersection of tangents becomes

$$amm'-a=h,\quad a(m+m')=k,$$

and the intersection of normals is

$$x=a(m^2+mm'+m'^2)+a=a\left(\frac{k^2}{a^2}-\frac{h+a}{a}\right)+a=\frac{k^2}{a}-h,$$

$$y=-amm'(m+m')=-a\,.\frac{h+a}{a}\frac{k}{a}=-\frac{k(h+a)}{a}.$$

Eliminating a, we obtain the locus.

646. [R. 4. c.] *A triangle of rods smoothly jointed at A, B, C is hung up by the corner A. Prove that the action at the joint C is inclined to the horizontal at an angle*

$$tan^{-1}\frac{(2W_1+W_3)\,cot\,\phi-W_2\,cot\,\theta}{2W_1+W_2+W_3},$$

where W_1, W_2, W_3 are the weights of BC, CA, AB and θ, ϕ are the inclinations of AB, AC to the vertical.

Take ac, cb, ba' on the same vertical to represent the weights of the three rods. Bisect these lines in 2, 1, 3 respectively.

Draw $2O$, $1O$ $\parallel$ to AC, BC. Then cO must be the direction of the reaction at C, and $3O$ must be $\parallel$ to AB.

Let cO make an angle ψ with the vertical, and let p be the perpendicular from O on aba'. Then

$$2c=p(\cot\phi-\cot\psi),\qquad c3=p(\cot\psi+\cot\theta)\,;$$

$$\therefore\ \frac{\cot\phi-\cot\psi}{\cot\psi+\cot\theta}=\frac{2c}{c3}=\frac{\frac{1}{2}W_2}{W_1+\frac{1}{2}W_3},$$

leading to the given value for $\cot\psi$.

647. [R. 9. a.] *Two rods AB, AC, each of weight W' and length $2a$, are rigidly connected so that they are at right angles and can turn freely in a vertical plane about a pivot at A. Two small rough rings, each of weight W, are placed one on each rod and are connected by a light string of length $2l$, passing over a smooth pulley at A. If μ be the coefficient of friction, prove that the distances of the rings from A must lie between*

$$l\pm\mu\left(l+\frac{W'}{W}a\right),$$

provided $\mu<\dfrac{Wl}{Wl+W'a}$.

Suppose that in a limiting position the ring on AB is about to slip down, and is distant x from A. Let R be the pressure on the rod, T the tension. Then, for the ring,

$$R=W\sin\theta,\qquad T+\mu R=W\cos\theta\,;\qquad\therefore\ T=W(\cos\theta-\mu\sin\theta).$$

So, from the other ring,

$$T=W(\sin\theta+\mu\cos\theta)\,;\qquad\therefore\ \tan\theta=\frac{1-\mu}{1+\mu}.$$

Now the C. of G. of the whole system is vertically below A.

Hence $\qquad W'.a\sin\theta+W.x\sin\theta=W'.a\cos\theta+W(2l-x)\cos\theta\,;$

$$\therefore\ \frac{1-\mu}{1+\mu}=\frac{W'.a+W.(2l-x)}{W'.a+W.x}\,;$$

$$\therefore\ x=l+\mu\left(l+\frac{W'}{W}.a\right).$$

Changing the sign of μ, we get the other extreme value, and this is supposed positive, implying the given limitation on the value of μ.

648. [R. 7. b. γ.] *A rocket fired vertically upwards bursts at its highest point h feet above the ground. If each fragment starts with the same velocity U, prove that all the fragments on reaching the ground lie within a circle of radius*

$$\frac{U}{g}\cdot\sqrt{U^2+2gh}.$$

If the shell bursts at S, the enveloping parabola has focus S, and latus-rectum $4h'$, where $U^2=2gh'$.

If it meets the ground in Q, and AS meets the ground in N, then

$$\begin{aligned} QN^2&=4AS\,.\,AN\\ &=4h'(h+h')\\ &=\frac{2U^2}{g}\left(h+\frac{U^2}{2g}\right)=\frac{U^2}{g^2}(2gh+U^2), \end{aligned}$$

and all the fragments must lie within a circle, centre N and radius QN.

649. [R. 2. a.] *A straight line PQ is drawn parallel to AB to meet the circumcircle of the triangle ABC in P and Q. Shew that the pedal lines of P and Q intersect on the perpendicular from C on AB.*

Let O be the orthocentre, and let CO meet the circumcircle in E, and EP, EQ meet AB in K, L and the pedal lines of P, Q respectively in U, V. Then OK, OL are parallel to the pedal lines and U, V are the middle points of PK, QL. $\therefore$ UV is parallel to KL. Hence, evidently, from similar triangles, the pedal lines must intersect on EO produced.

650. [L[1]. 3. b.] *The tangent at a fixed point P of an ellipse whose foci are S and S' meets a pair of conjugate diameters in T and T'. Shew that the locus of the other intersection of the circles SPT, $S'PT'$ is a circle.*

Produce SP to H, making $PH=PS'$. The remaining tangents TQ, $T'Q'$ to the ellipse must be parallel, since PQ, PQ' are parallel to the conjugate diameters, and $\therefore$ QQ' must be a diameter.

It is evident from the figure that the triangles HTT' and $S'TT'$ are equal in all respects. $\therefore$ $\angle HTT'=S'TT'=STQ$.

$\therefore$ $\angle HTS=T'TQ$. Similarly $HT'S=TT'Q'$.

Hence S, T, H, T' are cyclic.

Now let the circles intersect again in P'.

Then $\angle SP'P=180^\circ-STP=180^\circ-SHT'=180^\circ-PS'T'$.

Similarly, $S'P'P=180^\circ-PST=180^\circ-PT'S'$;

$\therefore$ $SP'S'=PS'T'+PT'S'=180^\circ-S'PT'$,

and is $\therefore$ constant. Hence the locus of P' is a circle through S and S'.

651. [A. 1. b.] *Prove that if a, b, c, x, y, z are rational, and*

$$a+b+c=0,\quad x+y+z=0,$$

the area of the triangle whose sides are

$$\sqrt{a^2+x^2},\quad \sqrt{b^2+y^2},\quad \sqrt{c^2+z^2}$$

is rational and equal to

$$\tfrac{1}{2}\{-(bcx^2+cay^2+abz^2)\}^{\frac{1}{2}}.$$

If Δ is the area, $16\Delta^2=2\Sigma(b^2+y^2)(c^2+z^2)-\Sigma(a^2+x^2)^2$.

Now, by data, $2\Sigma b^2c^2-\Sigma a^4=0$ and $2\Sigma y^2z^2-\Sigma x^4=0$.

Hence $$\begin{aligned}16\Delta^2&=2\Sigma(b^2z^2+c^2y^2)-2\Sigma a^2x^2\\ &=2\Sigma(b^2+c^2-a^2)x^2.\end{aligned}$$

Now $a+b+c=0$; $\therefore\ (b+c)^2=a^2$, *i.e.* $b^2+c^2-a^2=-2bc$;

$$\therefore\ 4\Delta^2=-\Sigma bcx^2.$$

Also

$$\begin{aligned}-\Sigma bcx^2&=-bcx^2-cay^2-ab(x+y)^2\\&=-b(c+a)x^2-a(b+c)y^2-2abxy\\&=b^2x^2+a^2y^2-2abxy=(bx-ay)^2,\end{aligned}$$

and $\therefore\ \Delta$ is rational.

652. [L[1]. 10. a.] *The polars of any three points with respect to the parabola $y^2=4ax$ form a triangle of area Δ_1: the tangents parallel to them form a triangle of area Δ_2, and Δ is the area of the triangle ABC. Shew that*

$$4\Delta_1\Delta_2=\Delta^2.$$

The polar of (x_1,y_1) is $\quad 2ax-yy_1+2ax_1=0,$

and the parallel tangent is $\quad 2ax-yy_1+\frac{1}{2}y_1^2=0.$

$$\therefore\ \Delta_1=\frac{\begin{vmatrix}2a, & -y_1, & 2ax_1\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\end{vmatrix}}{\Pi\begin{vmatrix}2a, & -y_1\\ 2a, & -y_2\end{vmatrix}}=\frac{16a^4\Delta^2}{8a^3\,.\,\Pi(y_1-y_2)}$$

and

$$\Delta_2=\frac{\begin{vmatrix}2a, & -y_1, & \frac{1}{2}y_1^2\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\end{vmatrix}^2}{\Pi\begin{vmatrix}2a, & -y_1\\ 2a, & -y_2\end{vmatrix}}=\frac{a^2\begin{vmatrix}1, & y_1, & y_1^2\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\end{vmatrix}^2}{8a^3\,.\,\Pi(y_1-y_2)}.$$

Now the determinant in the numerator is $\Pi(y_1-y_2)$;

$$\therefore\ \Delta_2=\frac{1}{8a}\,.\,\Pi(y_1-y_2);$$

$$\therefore\ \Delta_1\Delta_2=\tfrac{1}{4}\Delta_2.$$

653. [R. 4. c.] *Three equal uniform rods of length l and weight w are smoothly jointed together to form a triangle ABC. This triangle is hung up by the joint A, and by two strings each of length $\frac{l}{\sqrt{2}}$ a weight W is attached to B and C. If the system hangs under gravity, shew that the thrust along BC is equal to*

$$\frac{1}{\sqrt{3}}\left\{w+\frac{W}{2}(1+\sqrt{3})\right\}.$$

Each of the strings makes an angle 45° with the horizontal. Hence, if X be the thrust in BC, we have, taking moments about A for AB,

$$X\,.\,l\sin 60^\circ=T\,.\,l\sin 75^\circ+w\,.\,\frac{l}{2}\cos 60^\circ+Y\,.\,l\cos 60^\circ,$$

where Y is the vertical component of the reaction at B.

Also for W, $2T\cos 45^\circ=W$, and for BC, $2y=w$;

$$\therefore\ X\,.\,l\frac{\sqrt{3}}{2}=\frac{W}{\sqrt{2}}\,.\,l\,.\,\frac{\sqrt{3}+1}{2\sqrt{2}}+w\,.\,\frac{l}{4}+\frac{w}{2}\,.\,\frac{l}{2},$$

whence

$$X=W\,.\,\frac{\sqrt{3}+1}{2\sqrt{3}}+\frac{w}{\sqrt{3}}.$$

654. [R. 7. a.] *A wedge of mass m_1 and angle α lies on a horizontal table, and a second wedge of the same angle and mass m_2 is placed upon it so that the upper face is horizontal. Upon this face is placed a particle of mass m_3. Shew that, in the ensuing motion, the total weight will exceed the pressure on the table by*

$$\frac{(m_1+m_2)(m_2+m_3)^2 g \sin^2\alpha}{(m_1+m_2)(m_2+m_3)\sin^2\alpha+m_1 m_2 \cos^2\alpha}.$$

Let f be the acceleration of m_1 horizontally.

Let f_1, f_2 be the accelerations of m_2 parallel and $\perp^r$ to the face of m_1, and f_3 be the acceleration of m_3 vertically downwards.

R the pressure between m_1 and m_2,

R' .... m_2 and m_3.

Then the equations are

$$m_3 f_3 = m_3 g - R'. \text{ ....(i)} \qquad m_2 f_1 = (m_2 g + R')\sin\alpha. \text{ ...(ii)}$$

$$m_2 f_2 = (m_2 g + R')\cos\alpha - R. \text{ ......(iii)} \qquad m_1 f = R\sin\alpha. \text{ ....(iv)}$$

$$f_2 = f\sin\alpha. \text{ ....(v)} \qquad f_3 = f_1\sin\alpha + f_2\cos\alpha. \text{ ......(vi)}$$

From (i) and (vi), $\quad m_3(f_1\sin\alpha+f_2\cos\alpha)=m_3 g-R'$,

whence from (ii) and (iii), substituting for f_1 and f_2, we find

$$R'\left(\frac{1}{m_2}+\frac{1}{m_3}\right)=\frac{R\cos\alpha}{m_2}. \text{ ....(vii)}$$

Also from (iii), (iv) and (v),

$$\left(g+\frac{R'}{m_2}\right)\cos\alpha-\frac{R}{m_2}=\frac{R\sin^2\alpha}{m_1}. \text{ ....(viii)}$$

From (vii) and (viii), we find

$$R=\frac{m_1 m_2(m_2+m_3)}{(m_1+m_2)(m_2+m_3)\sin^2\alpha+m_1 m_2\cos^2\alpha}\cdot g\cos\alpha.$$

Now the pressure on the table is $R\cos\alpha+m_1 g$, and therefore the quantity required is $(m_2+m_3)g-R\cos\alpha$, which, substituting for R, reduces to the given expression.

655. [R. 7. b. γ.] *An elastic particle is projected with velocity V from a point on the ground and strikes a smooth vertical wall with its foot at a distance a from the point of projection. Prove that after rebounding from the wall, the particle can strike the ground at a point further from the wall than the point of projection if*

$$V^2 > \frac{1+e}{e}\cdot ga.$$

Let α be the elevation, t and t' the times of going and returning. Then $V\cos\alpha\,.\,t=a$.

After impact the horizontal velocity is $eV\cos\alpha$, while the vertical velocity is unchanged. Hence since the vertical distance described in time $t+t'$ is zero, we have

$$V\sin\alpha(t+t')-\tfrac{1}{2}g(t+t')^2=0,$$

whence $t+t'=\dfrac{2V\sin\alpha}{g}$. Hence if the particle strikes the ground again at distance x from the wall,

$$x=eV\cos\alpha\,.\,t'=eV\cos\alpha\left(\frac{2V\sin\alpha}{g}-\frac{a}{V\cos\alpha}\right)$$

$$=e\,.\,\frac{V^2\sin 2\alpha}{g}-ae.$$

Hence, if $x > a$, we have $\dfrac{V^2 \sin 2a}{g} > \dfrac{1+e}{e} . a$,

and since $\sin 2a \ngtr 1$, this is *possible* if the given condition is satisfied.

656. [K. 10. e.] *The angles APB, AQB subtended at two variable points P, Q by two fixed points A, B differ by a constant angle, and the ratios $AP : BP$ and $AQ : BQ$ are proportionals. Shew that if P describe a circle, Q will describe either a circle or a straight line.*

On AB describe an isosceles triangle ADB, having its vertical angle equal to the given constant angle. Draw any line through D and take points on it such that $DP . DQ = DA^2$, P and Q being supposed on opposite sides of AB. Then DA, DB touch the circles PAQ, PBQ respectively;

$$\therefore \angle DAQ = APD, \text{ and } DBQ = BPD.$$

Hence, adding, $AQB - ADB = APB$, *i.e.* $AQB - APB =$ the given angle. Further, by similar triangles,

$$AP : PD = AQ : AD, \; i.e. \; AP : AQ = PD : AD.$$

Similarly, $\quad BP : BQ = PD : BD; \quad \therefore AP : AQ = BP : BQ,$

$$i.e. \; AP : BP = AQ : BQ.$$

Hence the points P and Q are related as in question, and since they are inverse points with respect to D, if P describe a circle, Q will describe either a circle or a straight line.

657. [L^1. 17. e.] *A, B, C, D, E are five points on a circle. Conics are drawn having E for focus and touching the sides of BCD, CDA, DAB, ABC respectively. Prove that they are parabolas and that their directrices are tangents to another parabola having E for focus.*

Reciprocating, we get a parabola, focus E, and four tangents, while the four conics become the circles circumscribing the triangles formed by these four, and since these circles pass through the focus E, the original conics must be parabolas.

Further, the centres of the four circles lie on a circle through E. This may be proved thus:

Let PQR be the triangle formed by three of the tangents, and let the other tangent meet its sides in P', Q', R'.

Denote the centres of the circles PQR, $PQ'R'$, $QP'R'$, $Q'P'R$, by α, β, γ, δ respectively.

Then $\quad \alpha P = \alpha E$ and $\beta P = \beta E$; $\quad \therefore \angle \alpha E \beta = \alpha P \beta = \alpha PQ - \beta PQ$.

But $\quad \alpha PQ = 90^\circ - PRQ$, $\beta PQ = 90^\circ - PQ'R'$; $\quad \therefore \alpha E\beta = Q'P'R$.

Again, since the line joining the centres of two circles is perpendicular to their common chord; $\quad \therefore \alpha\gamma\beta = QER' = QP'R'$.

$\therefore \alpha E\beta = \alpha\gamma\beta$, *i.e.* α, β, γ, E are cyclic. Similarly, δ lies on the same circle. The theorem in question is the reciprocal of this.

658. [A. 3.] *If the system of equations*

$$x+y+z=0, \quad ax^2+by^2+cz^2=0, \quad ax^4+by^4+cz^4=0$$

admit of a solution other than $x=y=z=0$, then

$$(b+c)(c+a)(a+b)\{(b+c)(c+a)(a+b)-8abc\}=0.$$

From the last two equations

$$\frac{a}{y^2z^2(y^2-z^2)} = \ldots = \ldots .$$

But $y+z=-x$; $\quad\therefore\ \dfrac{a}{yz(y-z)}=\ldots=\ldots$

$$=\frac{b+c}{x(y-z)(x-y-z)}=\frac{b+c}{2x^2(y-z)}=\ldots.$$

Hence, if x, y, z are not all equal, $\dfrac{b+c}{a}=\dfrac{2x^2}{yz}$, etc.

These are satisfied if $b+c=0$, $x=0$, and sets of similar conditions. Otherwise, multiplying, we have $(b+c)(c+a)(a+b)=8abc$.

659. [K. 2. b. c.] *If t_1, t_2, t_3 be the tangents from the centre of the nine-point circle of a triangle to the escribed circles, prove that*

$$\frac{t_1^2}{r_1}+\frac{t_2^2}{r_2}+\frac{t_3^2}{r_3}=\frac{R+12r}{16\,sin\,\frac{A}{2}\,sin\,\frac{B}{2}\,sin\,\frac{C}{2}}.$$

We have $\quad t_1^2=(\tfrac{1}{2}R+r_1)^2-r_1^2=\tfrac{1}{4}R^2+Rr_1$;

$$\therefore\ \Sigma\frac{t_1^2}{r_1}=\frac{1}{4}R^2.\Sigma\frac{1}{r_1}+3R=\frac{1}{4}R^2.\frac{1}{r}+3R$$

$$=\frac{R^2+12Rr}{4r}$$

and $\quad r=4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}.$

660. [K. 20. c.] *Prove that if s be the sum of the four values of θ less than 2π which satisfy the equation*

$$a\cos 2(\theta-\alpha)+b\cos(\theta-\beta)+c=0,$$

then $s=2n\pi+4\alpha$, where n is some integer.

Putting $\tan\dfrac{\theta}{2}=t$, we have $\sin\theta=\dfrac{2t}{1+t^2}$, $\cos\theta=\dfrac{1-t^2}{1+t^2}$,

$$\sin 2\theta=\frac{4t(1-t^2)}{(1+t^2)^2},\quad \cos 2\theta=\frac{1-6t^2+t^4}{(1+t^2)^2};$$

whence substituting and reducing, we find

$$t^4(a\cos 2\alpha-b\cos\beta+c)-2t^3(2a\sin 2\alpha-b\sin\beta)$$
$$+2t^2(c-3a\cos 2\alpha)+2t(2a\sin 2\alpha+b\sin\beta)$$
$$+a\cos 2\alpha+b\cos\beta+c=0.$$

If the roots of this equation be t_1, t_2, etc., then

$$\tan\frac{s}{2}=\frac{\Sigma t_1-\Sigma t_1t_2t_3}{1-\Sigma t_1t_2+t_1t_2t_3t_4}$$

$$=\frac{8a\sin 2\alpha}{(a\cos 2\alpha-b\cos\beta+c)-2(c-3a\cos 2\alpha)+(a\cos 2\alpha+b\cos\beta+c)}$$

$=\tan 2\alpha$; $\quad\therefore\ \dfrac{s}{2}=n\pi+2\alpha.$

661. [L¹. 17. e.] *Prove that if the conics*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\quad and\quad \alpha x^2+\beta y^2+2\gamma xy=1$$

intersect at right angles, then

$$\frac{\alpha a^2-1}{\alpha a^2b^2}=\frac{2}{a^2+b^2}=\frac{\beta b^2-1}{\beta a^2b^2}.$$

Suppose the conics intersect at a point whose eccentric angle is θ. The tangents are

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,\ x(\alpha a\cos\theta+\gamma b\sin\theta)+y(\gamma a\cos\theta+\beta b\sin\theta)=1.$$

If these are perpendicular, then

$$b(\alpha a\cos\theta+\gamma b\sin\theta)\cos\theta+a(\gamma a\cos\theta+\beta b\sin\theta)\sin\theta=0$$

or
$$ab(\alpha\cos^2\theta+\beta\sin^2\theta)+\gamma(a^2+b^2)\sin\theta\cos\theta=0. \quad\text{.........(i)}$$

Also, since the point θ is on the second conic,

$$\therefore\ \alpha a^2\cos^2\theta+\beta b^2\sin^2\theta+2\gamma\,.\,ab\sin\theta\cos\theta=1=\cos^2\theta+\sin^2\theta,$$

$$i.e.\ (\alpha a^2-1)\cos^2\theta+(\beta b^2-1)\sin^2\theta+2\gamma\,.\,ab\sin\theta\cos\theta=0. \quad\text{............(ii)}$$

The equations (i) and (ii) to determine $\tan\theta$ must be identical. Hence, comparing coefficients, we have

$$\frac{\alpha a^2-1}{ab\alpha}=\frac{\beta b^2-1}{ab\beta}=\frac{2ab}{a^2+b^2}.$$

662. [R. 1. e.] *A hexagonal framework is made of six light smoothly-jointed wires, and has six equal particles attached to its angular points. One of its sides is maintained in a fixed horizontal position and the other sides are held in the same straight line with it and then released. Shew that when the framework is a regular hexagon the velocity of separation of the two horizontal sides will be*

$$\left\{\frac{3\sqrt{3}}{2}ag\right\}^{\frac{1}{2}},$$

where a is the length of a side.

Let v be the velocity of either of the upper particles, when the rods make an angle θ with the vertical. Then the velocity of either of the lower particles is $2v\cos\theta$ vertically.

Hence the equation of energy is

$$mv^2+m(2v\cos\theta)^2=2mg\,.\,a\sin\theta+2mg\,.\,2a\sin\theta,$$

$$i.e.\ v^2(1+4\cos^2\theta)=6ga\sin\theta.$$

Hence, when $\theta=60°$, $2v^2=3\sqrt{3}ga$, and the required velocity is

$$2v\cos 60°=v=\left\{\frac{3\sqrt{2}}{2}ga\right\}^{\frac{1}{2}}.$$

663. [R. 9. b.] *A ball A of mass nm impinges obliquely on another B, of mass m, at rest. Shew that there will in general be two positions in which B can be placed, so that the direction of A's motion may be turned through a given angle α, if*

$$n<\frac{1+e}{2}\operatorname{cosec}\alpha-\frac{1-e}{2}.$$

If, however, there be only one position for B, and if $e=1$, $\alpha=\frac{\pi}{3}$, shew that $n=\frac{2}{\sqrt{3}}$ and the velocities of A before and after impact, and that of B, will be as

$$\cos\frac{\pi}{12}:\sin\frac{\pi}{12}:1.$$

With the usual notation, the equations for impact are

$nv\cos\theta+v'\cos\phi=nu\cos\beta$, ...(i) $\quad v\cos\theta-v'\cos\phi=-eu\cos\beta$, ...(ii)

$v\sin\theta=u\sin\beta$, ...(iii) $\quad v'\sin\phi=0$. ....(iv)

From (i) and (ii), $(n+1)v\cos\theta=(n-e)u\cos\beta$;

$\therefore$ from (iii), $$\tan\theta=\frac{n+1}{n-e}\cdot\tan\beta.$$

By the question $\theta=\alpha+\beta$, and we $\therefore$ have

$$(n-e)(\tan\alpha+\tan\beta)=(n+1)\tan\beta(1-\tan\alpha\tan\beta),$$

$$\textit{i.e.}\ (n+1)\tan\alpha\tan^2\beta-(1+e)\tan\beta+(n-e)\tan\alpha=0.$$

This will give two values of β, provided the roots are real, *i.e.* if

$$(1+e)^2>4(n+1)(n-e)\tan^2\alpha,$$

$$\textit{i.e.}\ (1+e)^2\operatorname{cosec}^2\alpha>(1+e)^2+4(n+1)(n-e)>(2n+1-e)^2,$$

$$\textit{i.e.}\ n<\frac{1+e}{2}\operatorname{cosec}\alpha-\frac{1-e}{2}.$$

If the quadratic has equal roots, this is an equality, and if $e=1$, $\alpha=\frac{\pi}{3}$, we have $n=\frac{2}{\sqrt{3}}$.

In this case $$\tan\beta=\tfrac{1}{2}\cdot\frac{1+e}{(n+1)\tan\alpha}=\frac{1}{2+\sqrt{3}};\quad\therefore\ \beta=\frac{\pi}{12},$$

and $$\tan\theta=\frac{\frac{2}{\sqrt{3}}+1}{\frac{2}{\sqrt{3}}-1}\cdot\tan\frac{\pi}{12}=2+\sqrt{3};\quad\therefore\ \theta=\frac{5\pi}{12}.$$

Hence, returning to the original equations, we have

$$\frac{v}{\sin\beta}=\frac{u}{\sin\theta}=\frac{v'}{\sin\beta\cos\theta+e\sin\theta\cos\beta},$$

$$\textit{i.e.}\ \frac{v}{\sin\frac{\pi}{12}}=\frac{u}{\cos\frac{\pi}{12}}=v'.$$

664. [**K. 10. e.**] *The tangents TP, TP' to a circle are bisected in M, M', and the lines joining P, P' to any point Q on the circle cut MM' in R, R'. Shew that T, R, R', Q lie on a circle.*

Let TQ meet PP' in U, MM' in V and the circle in Q'.

Then $$\frac{VR}{VQ}=\frac{PU}{UQ}\ \text{and}\ \frac{VR'}{VQ}=\frac{UP'}{UQ};$$

$$\therefore\ \frac{VR\,.\,VR'}{VQ^2}=\frac{PU\,.\,UP'}{UQ^2}=\frac{Q'U\,.\,UQ}{UQ^2}=\frac{Q'U}{UQ}.$$

Also, since $(TU, Q'Q)$ is harmonic,

$$\therefore\ \frac{TQ}{TQ'}=\frac{QU}{UQ'}=\frac{TQ+QU}{TQ'+UQ'}=\frac{2VQ}{2VT};$$

$$\therefore\ \frac{VR\,.\,VR'}{VQ^2}=\frac{VQ}{VT},\quad\textit{i.e.}\ VR\,.\,VR'=VQ\,.\,VT;$$

$\therefore$ the points T, R', R, Q are cyclic.

665. [H. 2. b.] *Shew that if m be an integer prime to* 30, *then* m^4-1 *is divisible by* 240; *and that the necessary and sufficient condition that* m^2-1 *should be divisible by* 24 *and* m^2+1 *by* 10 *simultaneously is that m should be of one of the forms* $30k\pm 7$ *or* $30k\pm 13$, *where k is any integer.*

Since m^2+1 is to be divisible by 5, m must be of one of the forms $5p\pm 2$, and m being odd, p must be odd, say $2n+1$. Thus m must be of one of the forms $10n+7$ or $10n+3$.

Hence, putting $n=3k+k'(k'=0, 1, 2)$, all possible forms are included in $30k+10k'+7$ and $30k+10k'+3$.

In the first form, we must exclude the case $k'=2$, and in the second $k'=0$, each of which makes m a multiple of 3. The remaining forms are

$$30k+7,\quad 30k+17,\quad 30k+13,\quad 30k+23,\qquad i.e.\ 30k\pm 7,\quad 30k\pm 13.$$

For the first part, by Fermat's Theorem, $m^4-1=M(5)$.

Also $m^4=(m-1)(m+1)(m^2+1)$, and these numbers are all even, two of them being consecutive; therefore m^4-1 is divisible by $(2\times 4\times 2)$, *i.e.* 16. Also either $m-1$ or $m+1$ is $M(3)$;

$$\therefore\ m^4-1 \text{ is divisible by } 5\times 16\times 3=240.$$

666. [K. 1. c.] *If* ρ_1, ρ_2, ρ_3 *be the distances of any point in the plane of an equilateral triangle of side a from the angular points, prove that*

$$\Sigma\rho_2^2\rho_3^2-\Sigma\rho_1^4+a^2.\Sigma\rho_1^2-a^4=0.$$

If α, β, γ are the perpendiculars from the point on the sides, we have

$$\rho_1^2=\frac{\beta^2+\gamma^2+2\beta\gamma\cos\frac{\pi}{3}}{\sin^2\frac{\pi}{3}},$$

$$i.e.\ \beta^2+\gamma^2+\beta\gamma=\tfrac{3}{4}\rho_1^2 \text{ and two similar equations. ....(i)}$$

Also $\alpha+\beta+\gamma=\frac{\sqrt{3}}{2}a$, and therefore, subtracting two of the set (i), we get

$$\beta-\gamma=-\frac{\sqrt{3}}{2}\frac{\rho_2^2-\rho_3^2}{a};$$

whence combining with the remaining equation in (i),

$$\beta^2+\gamma^2=\frac{1}{2}\rho_1^2+\frac{1}{4}\frac{(\rho_2^2-\rho_3^2)^2}{a^2}$$

and
$$\beta\gamma=\frac{1}{4}\rho_1^2-\frac{1}{4}\frac{(\rho_2^2-\rho_3^2)^2}{a^2},$$

and two similar sets. But $\frac{1}{2}.\Sigma(\beta^2+\gamma^2)+2\Sigma\beta\gamma=\frac{3}{4}a^2$. Hence, substituting, we get

$$\frac{1}{4}.\Sigma\rho_1^2+\frac{1}{8a^2}.\Sigma(\rho_2^2-\rho_3^2)^2+\frac{1}{2}.\Sigma\rho_1^2-\frac{1}{2a^2}.\Sigma(\rho_2^2-\rho_3^2)^2=\frac{3}{4}a^2,$$

which reduces to the given form.

667. [K. 11. a.] *Shew that the limiting points of the system*

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+c')=0$$

subtend a right angle at the origin if

$$\frac{c}{g^2}+\frac{c'}{f^2}=2.$$

If the circle $x^2+y^2+2Gx+2Fy+C=0$ cuts the two given circles of the system orthogonally, we have

$$2Gg-C-c=0,\quad 2Ff-C-c'=0,$$

and the orthogonal circle is

$$x^2+y^2+\frac{C+c}{g}x+\frac{C+c'}{f}y+C=0. \quad \text{.........................(i)}$$

This circle passes through the limiting points. Hence, the line of centres being

$$\frac{x}{g}+\frac{y}{f}=-1, \quad \text{...(ii)}$$

the lines joining the origin to the limiting points, which are the intersections of (i) and (ii), are

$$x^2+y^2-\left(\frac{C+c}{g}x+\frac{C+c'}{f}\cdot y\right)\left(\frac{x}{g}+\frac{y}{f}\right)+C\left(\frac{x}{g}+\frac{y}{f}\right)^2=0,$$

and these are perpendicular if

$$2-\frac{C+c}{g^2}-\frac{C+c'}{f^2}+\frac{C}{g^2}+\frac{C}{f^2}=0,$$

$$i.e.\ \frac{c}{g^2}+\frac{c'}{f^2}=2.$$

668. [L¹. 5. c.] *Shew that the sum of the squares of the normals from* (ξ, η) *to* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is*

$$2\left\{a^2+b^2+\frac{a^2-2b^2}{a^2-b^2}\cdot\xi^2+\frac{b^2-2a^2}{b^2-a^2}\cdot\eta^2\right\}.$$

The eccentric angles of the feet of the normals satisfy the equation

$$a\xi\sin\theta-b\eta\cos\theta=(a^2-b^2)\sin\theta\cos\theta$$

or $(a^2-b^2)^2\cos^4\theta-2a\xi(a^2-b^2)\cos^3\theta+[a^2\xi^2+b^2\eta^2-(a^2-b^2)^2]\cos^2\theta+\ldots+\ldots=0,$

whence $\quad \Sigma\cos\alpha=\frac{2a\xi}{a^2-b^2},\ \Sigma\cos^2\alpha=\frac{2(a^2-b^2)^2+2(a^2\xi^2-b^2\eta^2)}{(a^2-b^2)^2};$

$$\therefore\ \Sigma(\xi-a\cos\alpha)^2=4\xi^2-2a\xi\cdot\frac{2a\xi}{a^2-b^2}+a^2\cdot\frac{2[a^2\xi^2-b^2\eta^2+(a^2-b^2)^2]}{(a^2-b^2)^2}.$$

Similarly,

$$\Sigma(\eta-b\sin\alpha)^2=4\eta^2-2b\eta\cdot\frac{2b\eta}{b^2-a^2}+b^2\cdot\frac{2[b^2\eta^2-a^2\xi^2+(b^2-a^2)^2]}{(b^2-a^2)^2}.$$

Adding these, the result follows.

669. [R. 7. b. γ.] *A particle is projected from any point, and at the same time an equal particle is let fall from a point on the directrix of its path. If the particles meet and coalesce, shew that the tangent to the new path at the point of union is at right angles to the original direction of projection, and the height of the directrix above the original point of projection is three-quarters of the height of the directrix of the first path.*

Suppose the particles coalesce after time t. The vertical distances described by them in this time are $u\sin\alpha\,.\,t-\frac{1}{2}gt^2$ and $\frac{1}{2}gt^2$. The sum of these must be equal to the distance of P, the point of projection from the directrix.

$$\therefore\ u\sin\alpha\,.\,t=\frac{u^2}{2g},\quad i.e.\ t=\frac{u}{2g\sin\alpha}.$$

If, after time t, the direction of motion makes an angle θ with the downward vertical, we have

$$-\cot\theta=\frac{u\sin\alpha-gt}{u\cos\alpha}=\tan\alpha-\operatorname{cosec}2\alpha=-\cot 2\alpha.$$

$\therefore\ \frac{\theta}{2}=\alpha$. But since the particles are equal, the new direction of motion makes an $\angle\frac{\theta}{2}$ with the vertical and is $\therefore$ at right angles to the original direction of projection.

Again, the height of the new directrix above the point of union is $\frac{\left(v\cos\frac{\theta}{2}\right)^2}{2g}$, where $v=gt=\frac{u}{2\sin\alpha}$, so that this distance is $\frac{1}{8}\frac{u^2}{g}\cot^2\alpha$.

Also the height of the point of union above the point of projection is

$$u\sin\alpha\,.\,t-\tfrac{1}{2}gt^2=\frac{u^2}{2g}-\frac{1}{8}\frac{u^2}{g}\operatorname{cosec}^2\alpha.$$

Hence the height required is

$$\frac{u^2}{2g}-\frac{1}{8}\frac{u^2}{g}=\frac{3}{4}\left(\frac{u^2}{2g}\right).$$

670. [R. 9. b.] *A smooth sphere of mass m' is suspended from a fixed point by an inextensible string. Another smooth sphere of mass m falling vertically impinges on the first with a velocity u. Prove that the initial velocity of the first sphere is*

$$\frac{mu\cos\theta\sin\theta(1+e)}{m'+m\sin^2\theta},$$

where θ is the inclination of the line joining the centres of the spheres to the vertical at the moment of impact, and e is the coefficient of restitution.

Let v_1 be the velocity of m along the line of centres, v_2 that of m' horizontally.

The impulse of the blow on m is $m(u\cos\theta-v_1)$, along the line of centres. Hence, resolving horizontally,

$$m(u\cos\theta-v_1)\sin\theta=m'v_2. \quad\text{(i)}$$

Also, by Newton's Law,

$$v_1-v_2\sin\theta=-eu\cos\theta. \quad\text{(ii)}$$

Solving (i) and (ii), we find

$$v_2=\frac{mu\cos\theta\sin\theta(1+e)}{m'+m\sin^2\theta}.$$

671. [A. 1. b.] *The four quantities a, b, m, n being supposed positive, shew that unless a and b are equal,*

$$(ma+nb)^{m+n}>(m+n)^{m+n}a^mb^n.$$

First suppose m, n positive integers. Take m quantities each equal to a, and n each equal to b. Then since the A.M. of these is $>$ their G.M., we have

$$\frac{ma+nb}{m+n}>(a^mb^n)^{\frac{1}{m+n}},$$

which is equivalent to the given result.

If m and n are not positive integers, take k such that km, kn are integers. Then, by the above,

$$\frac{kma+knb}{km+kn}>(a^{km}b^{kn})^{\frac{1}{km+kn}},$$

from which k disappears.

672. [L^1. 17. a.] *Shew that the equation to the straight lines joining the origin to the points of intersection of the conics*

$$u_0+u_1+u_2=0, \quad v_0+v_1+v_2=0$$

(*where u_n, v_n are homogeneous functions of x and y of order n*) *is*

$$(u_0v_1-u_1v_0)(u_1v_2-u_2v_1)=(u_0v_2-u_2v_0)^2.$$

Hence find the condition that the conics

$$ax^2+by^2=gx, \quad a'x^2+b'y^2=f'y$$

may have contact of the second order.

Let U_r be the result of putting $y=mx$ in u_r and dividing by x^r. Then we shall have

$$u_0+U_1x+U_2x^2=0, \quad v_0+V_1x+V_2x^2=0,$$

and eliminating x from these, the equation to determine m is

$$(u_0V_1-U_1v_0)(U_1V_2-U_2V_1)=(u_0V_2-U_2v_0)^2,$$

and now putting $m=\dfrac{y}{x}$, and clearing of fractions, we obtain the equation to the lines in the form given.

If the conics are $u_1+u_2=0$, $v_1+v_2=0$, then the equation to the three chords from the origin to the other points of intersection is

$$u_1v_2-u_2v_1=0.$$

In the case of the given conics, this is

$$f'y(ax^2+by^2)=gx(a'x^2+b'y^2) \text{ or } ga'x^3-f'ax^2y+gb'xy^2-f'by^3=0.$$

If the conics have contact of the second order, these three lines coincide, and $\therefore$ the expression on the left is a perfect cube. The conditions for this may be given in a variety of forms, among others,

$$27g^2a'^2b=f'^2a^3 \quad \text{and} \quad 27f'^2a'b^2=g^2b'^3.$$

673. [L^1. 5. b.] *Prove that two parabolas can be drawn through the feet of the normals from* (h, k) *to* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, *and that their latera-recta are*

$$\frac{2a^2b^2(ah\pm bk)}{(a^2-b^2)(a^2+b^2)^{\frac{3}{2}}}.$$

The rectangular hyperbola through the feet of the normals is

$$(a^2-b^2)xy+b^2xk-a^2yh=0,$$

and $\therefore$ any conic through these four points is of the form

$$b^2x^2+a^2y^2-a^2b^2+\lambda[(a^2-b^2)xy+b^2xk-a^2yh]=0.$$

If this is a parabola

$$a^2b^2=\tfrac{1}{4}\lambda^2(a^2-b^2)^2, \quad i.e.\ \lambda=\pm\frac{2ab}{a^2-b^2},$$

and the equation takes the form

$$(a^2-b^2)(bx\pm ay)^2\pm 2ab(b^2xk-a^2yh)-a^2b^2(a^2-b^2)=0.$$

Now the latus-rectum of $(\alpha x+\beta y)^2+2gx+2fy+c=0$ is

$$\frac{2(f\alpha-g\beta)}{(\alpha^2+\beta^2)^{\frac{3}{2}}},$$

giving the required values in these cases.

674. [L[1]. 6. c.] *P and Q are points on the equilateral hyperbola $xy=k^2$, such that the osculating circle at P passes through Q. Shew that the locus of the pole of PQ is*

$$(x^2+y^2)^2=4k^2xy.$$

Let the equation to PQ be $px+qy=1$, P being $\left(\frac{k}{m}, km\right)$. The equation to the osculating circle is then of the form

$$xy-k^2+\lambda(m^2x+y-2km)(px+qy-1)=0,$$

with the conditions $m^2p=q$.

Also $p\frac{k}{m}+q\,.\,km=1$, whence $p=\frac{m}{k(1+m^4)}$, $q=\frac{m^3}{k(1+m^4)}$, so that PQ is

$$mx+m^3y-k(1+m^4)=0.$$

Comparing this with $xy'+x'y-2k^2=0$, we have

$$\frac{y'}{m}=\frac{x'}{m^3}=\frac{2k}{1+m^4},$$

whence, eliminating m, we obtain the required locus of (x', y').

675. [R. 7. b. γ.] *Heavy beads, n in number, are attached to a string at regular distances a apart, and lie heaped together on a table. One is raised to a height just less than a, and from that position is projected vertically upwards with velocity V. Shew that, if*

$$V^2=\tfrac{1}{3}ga(n-2)(2n^2+n+3),$$

the n^{th} bead will just not rise from the table.

Let u_r be the velocity of the system just before the r^{th} bead leaves the table, v_r just after. Then, since the total momentum is unaltered by the jerk which sets the r^{th} bead in motion,

$$\therefore\ (r-1)u_r=rv_r.$$

Also $u_r^2=v_{r-1}^2-2ga$, whence

$$(r-1)^2u_r^2-(r-2)^2\,.\,u_{r-1}^2=-2ga(r-1)^2.$$

Taking this equation for all values of r from n to 3, and adding, we obtain

$$(n-1)^2u_n^2-1^2\,.\,u_2^2=-2ga[(n-1)^2+(n-2)^2+\ldots+2^2]$$

$$=-2ga\left[\frac{n(n-1)(2n-1)}{6}-1\right].$$

But $u_2=V$, and the condition required is $u_n=0$, leading to

$$V^2=2ga\,.\,\frac{2n^3-3n^2+n-6}{6}=\tfrac{1}{3}ga(n-2)(2n^2+n+3).$$

676. [R. 7. b. γ.] *A heavy particle projected with velocity u strikes at an angle of 45° an inclined plane of angle β, which passes through the point of projection. Shew that the vertical height of the point struck above the point of projection is*

$$\frac{u^2}{g}\cdot\frac{1+\cot\beta}{2+2\cot\beta+\cot^2\beta}.$$

Suppose the particle strikes the plane after time t. Then we must have

$$u\cos(\alpha-\beta)-g\sin\beta\,.\,t=-[u\sin(\alpha-\beta)-g\cos\beta\,.\,t],$$

$$i.e.\ u[\cos(\alpha-\beta)+\sin(\alpha-\beta)]=gt(\cos\beta+\sin\beta).$$

672. [L^1. 17. a.] *Shew that the equation to the straight lines joining the origin to the points of intersection of the conics*

$$u_0+u_1+u_2=0, \quad v_0+v_1+v_2=0$$

(*where u_n, v_n are homogeneous functions of x and y of order n*) *is*

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$$a^2b^2=\tfrac{1}{4}\lambda^2(a^2-b^2)^2, \quad i.e. \ \lambda=\pm\frac{2ab}{a^2-b^2},$$

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giving the required values in these cases.

674. [L¹. 6. c.] *P and Q are points on the equilateral hyperbola $xy=k^2$, such that the osculating circle at P passes through Q. Shew that the locus of the pole of PQ is*

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Let the equation to PQ be $px+qy=1$, P being $\left(\frac{k}{m}, km\right)$. The equation to the osculating circle is then of the form

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with the conditions $m^2p=q$.

Also $p\frac{k}{m}+q\,.\,km=1$, whence $p=\frac{m}{k(1+m^4)}$, $q=\frac{m^3}{k(1+m^4)}$, so that PQ is

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Comparing this with $xy'+x'y-2k^2=0$, we have

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Also $u_r^2=v_{r-1}^2-2ga$, whence

$$(r-1)^2u_r^2-(r-2)^2\,.\,u_{r-1}^2=-2ga(r-1)^2.$$

Taking this equation for all values of r from n to 3, and adding, we obtain

$$(n-1)^2u_n^2-1^2\,.\,u_2^2=-2ga[(n-1)^2+(n-2)^2+\ldots+2^2]$$

$$=-2ga\left[\frac{n(n-1)(2n-1)}{6}-1\right].$$

But $u_2=V$, and the condition required is $u_n=0$, leading to

$$V^2=2ga\,.\,\frac{2n^3-3n^2+n-6}{6}=\tfrac{1}{3}ga(n-2)(2n^2+n+3).$$

676. [R. 7. b. γ.] *A heavy particle projected with velocity u strikes at an angle of 45° an inclined plane of angle β, which passes through the point of projection. Shew that the vertical height of the point struck above the point of projection is*

$$\frac{u^2}{g}\,.\,\frac{1+\cot\beta}{2+2\cot\beta+\cot^2\beta}.$$

Suppose the particle strikes the plane after time t. Then we must have

$$u\cos(\alpha-\beta)-g\sin\beta\,.\,t=-[u\sin(\alpha-\beta)-g\cos\beta\,.\,t],$$

$$i.e.\ u[\cos(\alpha-\beta)+\sin(\alpha-\beta)]=gt(\cos\beta+\sin\beta).$$

But $$t=\frac{2u\sin(\alpha-\beta)}{g\cos\beta};\ \therefore\ \cot(\alpha-\beta)=1+2\tan\beta,$$

$$\text{i.e. } \frac{\cos(\alpha-\beta)}{1+2\tan\beta}=\frac{\sin(\alpha-\beta)}{1}=\frac{1}{\sqrt{2+4\tan\beta+4\tan^2\beta}}=\frac{1}{\sqrt{k}}, \text{ say.}$$

Hence

$$\sin\alpha=\sin(\overline{\alpha-\beta}+\beta)=\frac{\cos\beta+(1+2\tan\beta)\sin\beta}{\sqrt{k}},\ t=\frac{2u}{g}\cdot\frac{1}{\sqrt{k}\cos\beta}.$$

Thus the vertical distance described in time t is

$$u\sin\alpha\,.\,t-\tfrac{1}{2}gt^2=\frac{2u^2}{g}\cdot\frac{1+\tan\beta+2\tan^2\beta}{k}-\frac{2u^2}{g}\cdot\frac{1+\tan^2\beta}{k}$$

$$=\frac{2u^2}{g}\cdot\frac{\tan\beta+\tan^2\beta}{k},$$

as given.

677. [K. 4.] *A triangle is given in species and one vertex is fixed while the others lie one on each of two given circles. Construct the triangle.*

Let A be the fixed vertex, and draw through A the diameter ADD' of the circle on which B lies. On AD describe a triangle ADE similar to ABC, and through D' draw $D'E'$ parallel to DE to meet AE produced in E'.

Then $$AD:AE=AB:AC,\ \text{i.e. } AD:AB=AE:AC,$$

and $\angle DAB=EAC$; $\therefore$ the triangles DAB, EAC are similar. So also are the triangles ABD', ACE'. Hence C lies on the circle on EE' as diameter. Hence the intersections of this circle with the second given circle are the possible positions of C.

678. [I. 1.] *A pure recurring decimal has N figures in its period, of which m are equal to a, n equal to b, etc. Prove that the sum of all the different decimals obtained by interchanging the figures in all possible ways is*

$$\frac{1}{9}\,\frac{(N-1)!}{m!\,n!\,p!\ldots}(ma+nb+pc+\ldots).$$

The number of permutations of the N figures among themselves is

$$\frac{N!}{m!\,n!\,p!\ldots}.$$

In $\dfrac{(N-1)!}{(m-1)!\,n!\,p!\ldots}$, of these the figure a will occupy the r^{th} place,

" $\dfrac{(N-1)!}{m!\,(n-1)!\,p!\ldots}$, ... b ....

and so on.

Hence the sum represented by the figures in the r^{th} place in all possible decimals is

$$\left\{\frac{(N-1)!}{(m-1)!\,n!\,p!\ldots}a+\frac{(N-1)!}{m!\,(n-1)!\,p!\ldots}b+\ldots\right\}\cdot\frac{1}{10^r}$$

$$=\frac{(N-1)!}{m!\,n!\,p!\ldots}(ma+nb+\ldots)\,.\,\frac{1}{10^r};$$

$\therefore$ the sum of all the decimals is

$$\frac{(N-1)!}{m!\,n!\,p!\ldots}(ma+nb+\ldots)\,.\sum_{r=1}^{r=\infty}\frac{1}{10^r}$$

and
$$\sum_{r=1}^{r=\infty}\frac{1}{10^r}=\frac{1}{9}.$$

679. [K. 2. a. b.] *Prove that the angle θ between the lines joining the orthocentre to the centres of the inscribed and circumscribed circles of a triangle is given by*

$$\cos\theta=\frac{1+p+2q+8r}{2\sqrt{(1+8r)(1+p+q+2r)}},$$

where $\cos A$, $\cos B$, $\cos C$ *are the roots of the equation* $x^3+px^2+qx+r=0$.

With the usual notation $\cos\theta=\dfrac{OP^2+PI^2-OI^2}{2OP\,.\,PI}$.

Now denoting the radius of the incircle by ρ, we have

$$OP^2=R^2(1+8r),\quad PI^2=2\rho^2+4R^2r,\quad OI^2=R^2-2R\rho.$$

Also
$$\rho=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2};$$

$$\therefore\ \rho^2=2R^2(1-\cos A)(1-\cos B)(1-\cos C)=2R^2(1+p+q+r),$$

whence $PI^2=4R^2(1+p+q+2r)$.

Further, $\rho=R(\cos A+\cos B+\cos C-1)=-R(p+1)$.

Making these substitutions, we obtain the result as given.

680. [L[1]. 2. b.] *If* (x_1y_1), (x_2y_2), (x_3y_3) *are the vertices of a triangle self-conjugate to the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, *shew that the area of the triangle is*

$$\frac{1}{2a^2b^2}(x_2y_3-x_3y_2)(x_3y_1-x_1y_3)(x_1y_2-x_2y_1).$$

The area of the triangle formed by the polars of the given points is

$$\Delta'=\frac{1}{2}\cdot\frac{\begin{vmatrix}\frac{x_1}{a^2}, & \frac{y_1}{b^2}, & 1\\ \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots\end{vmatrix}^2}{\begin{vmatrix}\frac{x_1}{a^2}, & \frac{y_1}{b^2}\\ \frac{x_2}{a^2}, & \frac{y_2}{b^2}\end{vmatrix}\begin{vmatrix}\ldots\\ \ldots\end{vmatrix}\begin{vmatrix}\ldots\\ \ldots\end{vmatrix}}$$

$$=\frac{1}{2}\cdot\frac{a^2b^2(2\Delta)^2}{\begin{vmatrix}x_1, & y_1\\ x_2, & y_2\end{vmatrix}\begin{vmatrix}\ldots\\ \ldots\end{vmatrix}\begin{vmatrix}\ldots\\ \ldots\end{vmatrix}},$$

where Δ is the area of the triangle formed by the three points. But if the triangle is self-conjugate, $\Delta'=\Delta$, whence the result.

681. [L[1]. 4. b. a.] *Shew that the envelope of the chord of the conic*

$$ax^2+2hxy+by^2+c=0,$$

the tangents at whose extremities cut at right angles, is the conic

$$(a^2+h^2)x^2+(b^2+h^2)y^2+2(a+b)hxy=\frac{(h^2-ab)c}{a+b}.$$

The equation to the director circle is

$$(ab-h^2)(x^2+y^2)+c(a+b)=0,$$

and the pole of $lx+my+1=0$ is $\dfrac{bcl-chm}{ab-h^2}, \dfrac{-chl+cam}{ab-h^2}$.

If this lies on the director circle, we have

$$c[(bl-hm)^2+(hl-am)^2]+(a+b)(ab-h^2)=0$$

or

$$(b^2+h^2)l^2-2h(a+b)m+(a^2+h^2)m^2+\lambda=0,$$

where $\lambda=\dfrac{(a+b)(ab-h^2)}{c}$, and the corresponding Cartesian equation is

$$(a^2+h^2)\lambda\,.\,x^2+2h(a+b)\lambda xy+(b^2+h^2)\lambda y^2+(ab-h^2)^2=0.$$

682. [R. 7. b. γ.] *A wire ABC in the form of an equilateral triangle is fixed on a horizontal table. A particle is projected from a point in BC in a direction parallel to BA. If the point of projection divides BC in the ratio* $2e:3e-1$, *prove that the particle will return to it after impinging on AC and AB.*

After the first impact at E let the direction of motion make an angle a with CA, and after the second impact at F, an angle β with AB. Then

$$\tan a=e\tan 60^\circ,\quad \tan\beta=e\tan(120^\circ-a),$$

whence

$$\tan\beta=\frac{\sqrt{3}e(1+e)}{3e-1}.$$

If the particle returns to P, the point of projection, we have

$$\frac{BP}{PF}=\frac{\sin\beta}{\sin 60^\circ},\quad \frac{PF}{PE}=\frac{\sin(120^\circ-a)}{\sin(120^\circ-a+\beta)}\ \text{and}\ PE=PC.$$

Hence

$$\frac{BP}{PC}=\frac{2}{\sqrt{3}}\,\frac{\sin\beta\sin(60^\circ+a)}{\sin(60^\circ+a-\beta)}$$

$$=\frac{2}{\sqrt{3}}\,\frac{\tan\beta(\sqrt{3}+\tan a)}{\sqrt{3}(1+\tan a\tan\beta)+(\tan a-\tan\beta)}.$$

Substituting and reducing, this fraction is

$$\frac{2e(1+e)^2}{3e^3+5e^2+e-1}=\frac{2e}{3e-1}.$$

E. M. Radford.

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